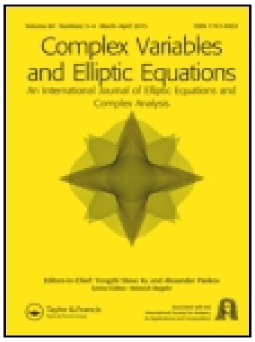

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The Bohr radius for starlike logharmonic mappings

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This paper studies the class consisting of univalent logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ in the unit disk U, where $h(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^k\right)$ and $g(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right)$ are analytic in U and $\varphi(z) = zh(z)/g(z)$ is a normalized starlike analytic function. A representation theorem for these mappings is obtained, which yields sharp distortion estimates, and a sharp Bohr radius.

Keywords: univalent logharmonic mappings; starlike logharmonic mappings; Bohr radius; distortion estimates

AMS Subject Classifications: Primary: 30C35; 30C45; Secondary: 35Q30

1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions defined in the unit disk $U := \{z : |z| < 1\}$. Bohr [1] in 1914 obtained the size of the moduli of the terms in the series expansion for an analytic self-map f of the unit disk U. This is now known as the Bohr inequality, which states if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(U)$ and $|f(z)| < 1$ in U, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ for all $|z| \leq 1/3$. In this instance, we say that the Bohr radius for the class of bounded analytic functions in the unit disk is $1/3$. Bohr in fact obtained the radius $1/6$. However, Wiener, Riesz and Schur (see [2–4]) independently established the sharp inequality for $|z| \leq 1/3$. Other proofs can also be found in [5–7].

The Bohr inequality can also be written in terms of its supremum norm, that is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $\|f\|_\infty = \sup_{|z|<1} |f(z)| < \infty$, then

$$\sum_{n=0}^{\infty} \left|a_n z^n\right| \leq \|f\|_\infty$$

when $|z| \leq 1/3$. Under this framework, Boas and Khavinson [8], and Aizenberg [9–12] have extended the inequality to several complex variables by finding the multidimensional Bohr radius. More recently, Defant et al. studied the link between the multidimensional

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Bohr radius and local Banach space theory,[13,14] and obtained the optimal asymptotic estimate for the n-dimensional Bohr radius on the polydisk U^n.[15]

The Bohr inequality has also emerged as an active area of research for operator algebraists after Dixon [16] used it to settle in the negative a conjecture that a Banach algebra satisfying a non-unital von Neumann inequality is necessarily an operator algebra. Subsequently, Paulsen and Singh [5], and Blasco [17] have extended the Bohr inequality in the context of Banach algebras.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$, an equivalent form of the Bohr inequality is

$$d\left(\sum_{n=0}^{\infty} \left|a_n z^n\right|, |a_0|\right) = \sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d(f(0), \partial U),$$

where d is the Euclidean distance. This form makes evident the notion of the Bohr phenomenon for analytic functions mapping the unit disk into a given domain. Let $S(\Omega)$ be the class consisting of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ from U into a domain Ω. The Bohr radius for Ω is the largest number $r_\Omega \in (0, 1)$ satisfying

$$d\left(\sum_{n=0}^{\infty} \left|a_n z^n\right|, |a_0|\right) = \sum_{n=1}^{\infty} \left|a_n z^n\right| \leq d(f(0), \partial\Omega)$$

for all $f \in S(\Omega)$ and $|z| < r_\Omega$.

If Ω is convex, it was shown by Aizenberg [18] that the sharp Bohr radius is $r_\Omega = 1/3$. This result includes the classical case $\Omega = U$. When Ω is any proper simply connected domain, Abu-Muhanna [19] showed that the best Bohr radius is $3 - 2\sqrt{2} \cong 0.17157$. In two recent papers,[20,21] the Bohr inequality was investigated for Ω being the exterior of a compact convex set or concave wedges. The Bohr radius for bounded harmonic functions was also obtained.

The aim of this paper is to extend the notion of the Bohr phenomenon to the context of starlike univalent logharmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$, where h and g are analytic in U. These logharmonic mappings are described in Section 2. Sharp distortion estimates are obtained in Section 3, and the final section finds the Bohr radius.

2. Logharmonic mappings

Let $\mathcal{B}(U)$ denote the set of functions $a \in \mathcal{H}(U)$ satisfying $|a(z)| < 1$ in U. A logharmonic mapping defined in U is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f},$$

where the second dilatation function a lies in $\mathcal{B}(U)$. Thus, the Jacobian

$$J_f = |f_z|^2 \left(1 - |a|^2\right)$$

is positive and all non-constant logharmonic mappings are therefore sense-preserving and open in U. In [22], the class of locally univalent logharmonic mappings is shown to play an instrumental role in validating the Iwaniec conjecture involving the Beurling–Ahlfors operator.

When f is a non-vanishing logharmonic mapping in U, it is known that f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{1}$$

where h and g are in $\mathcal{H}(U)$. In [23], Mao et al. introduced the Schwarzian derivative for these non-vanishing logharmonic mappings. They established the Schwarz lemma for this class and obtained two versions of Landau's theorem. Denote by $\mathscr{P}_{LH}$ the class consisting of logharmonic mappings f in U of the form (1) satisfying $\operatorname{Re} f(z) > 0$ for all $z \in U$. The subclass $\mathscr{P}_{LH(M)}$ defined by

$$\mathscr{P}_{LH(M)} = \left\{ f : f = h(z)\overline{g(z)} \in \mathscr{P}_{LH}, \left| \frac{h(z)}{g(z)} - M \right| < M, \; M \geq 1 \right\}$$

was recently investigated in [24].

If f is a non-constant logharmonic mapping of U which vanishes only at $z = 0$, then [25] f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{2}$$

where m is a non-negative integer, $\operatorname{Re}\beta > -1/2$, and h and g are analytic functions in U satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (2) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)} \frac{1 + a(0)}{1 - |a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping in U vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \quad z \in U,$$

where $\operatorname{Re}\beta > -1/2$, $0 \notin (hg)(U)$ and $g(0) = 1$. This class has been widely studied in the works of [25–29]. In this case, it follows that $F(\zeta) = \log f(e^{\zeta})$ are univalent harmonic mappings of the half-plane $\{\zeta : \operatorname{Re}(\zeta) < 0\}$. Studies on univalent harmonic mappings can be found in [30–37]. Such mappings are closely related to the theory of minimal surfaces (see [38,39]).

Denote by S_{Lh} the class consisting of univalent logharmonic maps f of the form

$$f(z) = zh(z)\overline{g(z)}$$

with the normalization $h(0) = g(0) = 1$. This paper gives emphasis to the subclass ST^0_{Lh} consisting of functions $f \in S_{Lh}$ which maps U onto a starlike domain (with respect to the origin). Thus, the linear segment joining the origin to every point $f(z)$ lies entirely in $f(U)$. Starlike logharmonic mappings is an active subject of investigation, and several recent works include those of [40–42].

3. Distortion theorem

Let $\mathcal{A}$ denote the class of analytic functions f in U normalized by the conditions $f(0) = 0 = f'(0) - 1$. Further let S^* be the class consisting of functions $f \in \mathcal{A}$ such that $f(U)$ is a starlike domain. We first establish an integral representation for starlike logharmonic mappings.

THEOREM A [43, Corollary 3.6] *Let p be analytic in U with $p(0) = 1$. Then* Re $p(z) > 0$ *in U if and only if there is a probability measure μ on ∂U such that*

$$p(z) = \int_{|x|=1} \frac{1 + xz}{1 - xz} \mathrm{d}\mu(x) \quad (|z| < 1).$$

THEOREM B [43, Theorem 3.9] *Let $f \in \mathcal{A}$. Then $f \in S^*$ if and only if there is a probability measure μ on ∂U so that*

$$\frac{zf'(z)}{f(z)} = \int_{|x|=1} \frac{1 + xz}{1 - xz} \mathrm{d}\mu(x) \quad (|z| < 1),$$

or equivalently,

$$f(z) = z \exp\left(\int_{|x|=1} -2\log(1 - xz)\mathrm{d}\mu(x)\right).$$

If $a \in \mathcal{B}(U)$, then $(1 + a(z))/(1 - a(z))$ has positive real part for $z \in U$, and the following result follows from Theorem A.

LEMMA 1 *If $a \in \mathcal{B}(U)$ with $a(0) = 0$, then*

$$\frac{a(z)}{1 - a(z)} = \int_{\partial U} \frac{xz}{1 - xz} \mathrm{d}\mu(x) \quad (|z| < 1)$$

for some probability measure μ on ∂U.

The following lemma establishes a link between starlike logharmonic functions and starlike analytic functions.

LEMMA 2 [28] *Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in U. Then $f \in ST^0_{Lh}$ if and only if $\varphi(z) = zh(z)/g(z) \in S^*$.*

THEOREM 1 *A logharmonic function $f(z) = zh(z)\overline{g(z)}$ belongs to ST^0_{Lh} if and only if there are two probability measures μ and ν on ∂U such that*

$$h(z) = \begin{cases} \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} - \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\right), & \text{if } \eta \neq \xi, \\ \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z} - \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\eta)\right), & \text{if } \eta = \xi, \end{cases} \tag{3}$$

and

$$g(z) = \begin{cases} \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} + \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\right), & \text{if } \eta \neq \xi, \\ \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z} + \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\eta)\right), & \text{if } \eta = \xi. \end{cases} \tag{4}$$

where $|\eta| = |\xi| = 1$.

Proof Let $\varphi(z) = zh(z)/g(z)$. Since

$$a(z) = \frac{zg'(z)/g(z)}{1 + zh'(z)/h(z)},$$

it follows that

$$\begin{aligned}\frac{z\varphi'(z)}{\varphi(z)} &= 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \\ &= \frac{zg'(z)}{g(z)}\left(\frac{1-a(z)}{a(z)}\right).\end{aligned}$$

Thus

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{1-a(s)} \cdot \frac{\varphi'(s)}{\varphi(s)} \mathrm{d}s\right) \tag{5}$$

and

$$h(z) = \frac{\varphi(z)}{z} g(z). \tag{6}$$

By Lemma 2 and Theorem B,

$$\frac{z\varphi'(z)}{\varphi(z)} = \int_{|\eta|=1} \frac{1+\eta z}{1-\eta z} \mathrm{d}\mu(\eta) \quad (|z| < 1), \tag{7}$$

or equivalently,

$$\varphi(z) = z \exp\left(\int_{|x|=1} -2\log(1-\eta z)\mathrm{d}\mu(\eta)\right). \tag{8}$$

From (5), (7) and Lemma 1 imply that g can be written as

$$g(z) = \exp\left(\int_0^z \int_{\partial U} \int_{\partial U} \frac{\xi}{1-\xi s} \cdot \frac{1+\eta s}{1-\eta s} \mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\mathrm{d}s\right)$$

for some probability measures μ and ν on U.

If $\eta \neq \xi$, then

$$\begin{aligned}g(z) &= \exp\left(\int_{\partial U}\int_{\partial U} \xi \int_0^z \left(\frac{2\eta}{(\eta-\xi)(1-\eta s)} - \frac{\eta+\xi}{(\eta-\xi)(1-\xi s)}\right) \mathrm{d}s\mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\right) \\ &= \exp\left(\int_{\partial U}\int_{\partial U} \left(-\frac{2\xi}{\eta-\xi}\log(1-\eta z) + \frac{\eta+\xi}{\eta-\xi}\log(1-\xi z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\right) \\ &= \exp\left(\int_{\partial U}\int_{\partial U} \left(\frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} + \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\xi)\right).\end{aligned}$$

On the other hand, if $\eta = \xi$, then

$$\begin{aligned}g(z) &= \exp\left(\int_{\partial U}\int_{\partial U}\int_0^z \frac{\eta+\eta^2 s}{(1-\eta s)^2}\mathrm{d}s\mathrm{d}\mu(\eta)\mathrm{d}\nu(\eta)\right) \\ &= \exp\left(\int_{\partial U}\int_{\partial U}\int_0^z \left(\frac{2\eta}{(1-\eta s)^2} + \frac{\eta^2 s-\eta}{(1-\eta s)^2}\right)\mathrm{d}s\mathrm{d}\mu(\eta)\mathrm{d}\nu(\eta)\right) \\ &= \exp\left(\int_{\partial U}\int_{\partial U}\left(\frac{2\eta z}{1-\eta z} + \log(1-\eta z)\right)\mathrm{d}\mu(\eta)\mathrm{d}\nu(\eta)\right).\end{aligned}$$

The representation for h follow from g by applying (6) and (8). □

Let

$$h_0(z) = \frac{1}{1-z}\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)z^n\right),$$

and

$$g_0(z) = (1-z)\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)z^n\right).$$

Then

$$f_0(z) = zh_0(z)\overline{g_0(z)} = \frac{z(1-\overline{z})}{1-z}\exp\left(\mathrm{Re}\left(\frac{4z}{1-z}\right)\right)$$

is the logharmonic Koebe function.

THEOREM 2 *Let* $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. *Then*

$$\frac{1}{1+|z|}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |h(z)| \le \frac{1}{1-|z|}\exp\left(\frac{2|z|}{1-|z|}\right), \tag{9}$$

$$(1+|z|)\exp\left(\frac{-2|z|}{1+|z|}\right) \le |g(z)| \le (1-|z|)\exp\left(\frac{2|z|}{1-|z|}\right), \tag{10}$$

$$|z|\exp\left(\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le |z|\exp\left(\frac{4|z|}{1-|z|}\right). \tag{11}$$

Equalities occur if and only if h, g, *and* f *are respectively appropriate rotations of* h_0, g_0 *and* f_0.

Proof Since $\varphi(z) = zh(z)/g(z) \in S^*$, it follows from (5) that

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{1-a(s)}\cdot\frac{\varphi'(s)}{\varphi(s)}\mathrm{d}s\right),$$

and thus

$$h(z) = \frac{\varphi(z)}{z}g(z), \quad \text{and} \quad f(z) = \varphi(z)|g(z)|^2.$$

For $|z| = r$, the known estimates

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \le \frac{1+r}{1-r},$$

$$\left|\frac{a(z)}{z(1-a(z))}\right| \le \frac{1}{1-r},$$

and

$$|\varphi(z)| \le \frac{r}{(1-r)^2}$$

yield

$$|g(z)| \le \exp\left(\int_0^r \frac{1}{1-s}\cdot\frac{1+s}{1-s}\mathrm{d}s\right) = (1-r)\exp\left(\frac{2r}{1-r}\right),$$

$$|h(z)| = \left|\frac{\varphi(z)}{z}g(z)\right| \le \frac{1}{(1-r)^2}\cdot(1-r)\exp\left(\frac{2r}{1-r}\right) = \frac{1}{1-r}\exp\left(\frac{2r}{1-r}\right),$$

and

$$|f(z)| = |\varphi(z)|\,|g(z)|^2 \le \frac{r}{(1-r)^2} \cdot (1-r)^2 \exp\left(\frac{4r}{1-r}\right) = r\exp\left(\frac{4r}{1-r}\right).$$

For the left estimates, (3) gives

$$\log|h(z)| = \mathrm{Re}\left(\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)\mathrm{d}\mu(\xi)\mathrm{d}\nu(\eta)\right), \quad |\eta| = |\xi| = 1,$$

where

$$K(z,\xi,\eta) = \begin{cases} \frac{\eta+\xi}{\eta-\xi}\log\frac{1-\xi z}{1-\eta z} - \log(1-\eta z), & \text{if } \eta \ne \xi; \\ \frac{2\eta z}{1-\eta z} - \log(1-\eta z), & \text{if } \eta = \xi. \end{cases}$$

Then for $|z| = r$,

$$\begin{aligned} \log|h(z)| &= \mathrm{Re}\left(\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)\mathrm{d}\mu(\xi)\mathrm{d}\nu(\eta)\right) \\ &\ge \min_{\mu,\nu}\left\{\min_{|z|=r}\mathrm{Re}\left(\int_{\partial U}\int_{\partial U} K(z,\xi,\eta)\mathrm{d}\mu(\xi)\mathrm{d}\nu(\eta)\right)\right\} \\ &= \min\left\{\min_{|z|=r}\inf_{0<|l|\le\pi/2}\left[-\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}(\eta z)}{1-(\eta z)}\right)\right] - \log(1+r),\right. \\ &\qquad \left.\frac{-2r}{1+r} - \log(1+r)\right\} \\ &= \min\left\{\inf_{0<|l|\le\pi/2}\min_{|z|=r}\left[-\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right] - \log(1+r),\right. \\ &\qquad \left.\frac{-2r}{1+r} - \log(1+r)\right\}, \end{aligned}$$

where $e^{2il} = \overline{\eta}\xi$.

Let

$$\Phi_r(l) = \begin{cases} \min_{|z|=r}\left[-\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right] - \log(1+r), & \text{if } 0 < |l| \le \pi/2; \\ \frac{-2r}{1+r} - \log(1+r), & \text{if } l = 0. \end{cases}$$

Since

$$\min_{|z|=r}\left[-\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right] = \min_{|z|=r}\mathrm{Re}\left[\frac{1+e^{2il}}{1-e^{2il}}\log\left(1+\frac{(1-e^{2il})z}{1-z}\right)\right],$$

evidently

$$\begin{aligned} &\lim_{l\to 0}\min_{|z|=r}\ \mathrm{Re}\left[\frac{1+e^{2il}}{1-e^{2il}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{(1-e^{2il})z}{1-z}\right)^k\right] \\ &= \min_{|z|=r}\mathrm{Re}\left[\frac{2z}{1-z} + \lim_{l\to 0}\left\{2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}(1-e^{2il})^{k-1}\left(\frac{z}{1-z}\right)^k\right\}\right] \\ &= \min_{|z|=r}\mathrm{Re}\left(\frac{2z}{1-z}\right) = -\frac{2r}{1+r}. \end{aligned}$$

Thus, $\Phi_r(l)$ is continuous in $|l| \le \pi/2$.

Moreover,

$$\begin{aligned}
\min_{|z|=r}\left[-\operatorname{Im}\left(\frac{1+e^{-2il}}{1-e^{-2il}}\right)\arg\left(\frac{1-e^{-2il}z}{1-z}\right)\right] &= \min_{|z|=r}\left[-\operatorname{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-z}{1-e^{-2il}z}\right)\right]\\
&= \min_{|z|=r}\left[-\operatorname{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}(e^{-2il}z)}{1-(e^{-2il}z)}\right)\right]\\
&= \min_{|w|=r}\left[-\operatorname{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}w}{1-w}\right)\right]
\end{aligned}$$

implies that $\Phi_r(l)$ is an even function in $|l| \le \pi/2$. Hence

$$\log|h(z)| \ge \inf_{0\le l\le\pi/2}\Phi_r(l).$$

Since

$$\max_{|z|=r}\arg\left(\frac{1-e^{2il}z}{1-z}\right) = 2\tan^{-1}\left(\frac{r\sin(l)}{1+r\cos(l)}\right),$$

this implies that

$$\log|h(z)| \ge \inf_{0\le l\le\pi/2} -2\cot(l)\tan^{-1}\left(\frac{r\sin(l)}{1+r\cos(l)}\right) - \log(1+r).$$

Evidently $\tan^{-1}(x) \le x$ for all $x \ge 0$, and so

$$\begin{aligned}
\log|h(z)| &\ge \inf_{0\le l\le\pi/2}\left(\frac{-2r\cos(l)}{1+r\cos(l)} - \log(1+r)\right)\\
&\ge \frac{-2r}{1+r} - \log(1+r).
\end{aligned}$$

For the lower bound of $|g(z)|$ in (10), a similar argument is applied to (4) which yields

$$\begin{aligned}
\log|g(z)| &\ge \inf_{0\le l\le\pi/2}\left(\frac{-2r\cos(l)}{1+r\cos(l)} + \log(1+r)\right)\\
&\ge \frac{-2r}{1+r} + \log(1+r).
\end{aligned}$$

Finally, it follows that

$$\begin{aligned}
|f(z)| = |z||h(z)||g(z)| &\ge \frac{r}{1+r}\exp\left(\frac{-2r}{1+r}\right)\cdot(1+r)\exp\left(\frac{-2r}{1+r}\right)\\
&= r\exp\left(\frac{-4r}{1+r}\right),
\end{aligned}$$

which establishes (11). □

Remark 1 The upper bounds for $|h(z)|$ and $|g(z)|$ in Theorem 2 were also obtained by Duman [44]. Here we not only established the sharp lower bounds, but also exhibit the extremal functions.

COROLLARY 1 *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Also, let $H(z) = zh(z)$ and $G(z) = zg(z)$. Then*

$$\frac{1}{2e} \leq d(0, \partial H(U)) \leq 1,$$
$$\frac{2}{e} \leq d(0, \partial G(U)) \leq 1,$$

and

$$\frac{1}{e^2} \leq d(0, \partial f(U)) \leq 1.$$

Equalities occur if and only if h, g and f are suitable rotations of h_0, g_0 and f_0, respectively.

Proof By (9),

$$d(0, \partial H(U)) = \liminf_{|z|\to 1} |H(z) - H(0)| = \liminf_{|z|\to 1} \frac{|H(z) - H(0)|}{|z|} \geq \frac{1}{2e}.$$

On the other hand, the minimum modulus principle shows that

$$d(0, \partial H(U)) = \liminf_{|z|\to 1} |H(z) - H(0)| = \liminf_{|z|\to 1} \frac{|H(z) - H(0)|}{|z|} \leq 1$$

since $|h(0)| = 1$. The same technique is applied to G and f to find the remaining inequalities. □

4. The Bohr radius for logharmonic mappings

Consider now logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ with

$$h(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^k\right) \quad \text{and} \quad g(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right).$$

THEOREM C [29, Theorem 3.3] *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then*

$$|a_n| \leq 2 + \frac{1}{n} \text{ and } |b_n| \leq 2 - \frac{1}{n}$$

for all $n \geq 1$. Equalities hold for f a rotation of the function f_0.

Our main results are the following theorems.

THEOREM 3 *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$, $H(z) = zh(z)$ and $G(z) = zg(z)$. Then*

(a)

$$|z| \exp\left(\sum_{n=1}^{\infty} |a_n||z|^n\right) \leq d(0, \partial H(U))$$

for $|z| \leq r_H \approx 0.1222$, where r_H is the unique root in $(0, 1)$ of

$$\frac{r}{1-r} \exp\left(\frac{2r}{1-r}\right) = \frac{1}{2e},$$

(b)

$$|z| \exp\left(\sum_{n=1}^{\infty} |b_n||z|^n\right) \leq d(0, \partial G(U))$$

for $|z| \leq r_G \approx 0.3659$, *where* r_G *is the unique root in* $(0, 1)$ *of*

$$r(1-r)\exp\left(\frac{2r}{1-r}\right) = \frac{2}{e}.$$

Both radii are sharp and are attained by appropriate rotations of $H_0(z) = zh_0(z)$ *and* $G_0(z) = zg_0(z)$, *respectively.*

Proof Note that

$$H(z) = z \exp\left(\sum_{k=1}^{\infty} a_k z^k\right) \quad \text{and} \quad G(z) = z \exp\left(\sum_{k=1}^{\infty} b_k z^k\right).$$

By Theorem C,

$$|a_n| \leq 2 + \frac{1}{n} \quad \text{and} \quad |b_n| \leq 2 - \frac{1}{n}$$

which are sharp bounds and Corollary 1 gives

$$d(0, \partial H(U)) \geq \frac{1}{2e} \quad \text{and} \quad d(0, \partial G(U)) \geq \frac{2}{e}.$$

Hence

$$\begin{aligned} r \exp\left(\sum_{n=1}^{\infty} |a_n| r^n\right) &\leq r \exp\left(\sum_{n=1}^{\infty} \left(2 + \frac{1}{n}\right) r^n\right) \\ &= r \exp\left(\frac{2r}{1-r} - \log(1-r)\right) \\ &\leq d(0, \partial H(U)) \end{aligned}$$

if and only if

$$\frac{r}{1-r} \exp \frac{2r}{1-r} \leq \frac{1}{2e}.$$

The Bohr radius, $r_H \approx 0.1222$ is therefore the positive solution of

$$\frac{r}{1-r} \exp \frac{2r}{1-r} = \frac{1}{2e}.$$

Likewise,

$$\begin{aligned} r \exp\left(\sum_{n=1}^{\infty} |b_n| r^n\right) &\leq r \exp\left(\sum_{n=1}^{\infty} \left(2 - \frac{1}{n}\right) r^n\right) \\ &= r \exp\left(\frac{2r}{1-r} + \log(1-r)\right) \\ &\leq d(0, \partial G(U)) \end{aligned}$$

if and only if

$$r(1-r)\exp\frac{2r}{1-r} \le \frac{2}{e}.$$

Hence the Bohr radius, r_G is the positive solution of

$$r(1-r)\exp\frac{2r}{1-r} = \frac{2}{e}$$

which gives $r_G \approx 0.3659$. Finally, it is evident that both radii are attained by suitable rotations of $H_0(z)$ and $G_0(z)$, respectively. □

THEOREM 4 *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then, for any real t,*

$$|z|\exp\left(\sum_{n=1}^{\infty}\left|a_n + e^{it}b_n\right| |z|^n\right) \le d(0, \partial f(U))$$

for $|z| \le r_0 \approx 0.09078$, where r_0 is the unique root in $(0, 1)$ of

$$r\exp\left(\frac{4r}{1-r}\right) = \frac{1}{e^2}.$$

The bound is sharp and is attained by a suitable rotation of the logharmonic Koebe function f_0.

Proof By Theorem C,

$$|a_n| \le 2 + \frac{1}{n} \text{ and } |b_n| \le 2 - \frac{1}{n}.$$

which are sharp bounds and Corollary 1 gives

$$d(0, \partial f(U)) \ge \frac{1}{e^2}$$

which is also sharp. Thus

$$\begin{aligned} r\exp\left(\sum_{n=1}^{\infty}|a_n|r^n + \sum_{n=1}^{\infty}|b_n|r^n\right) &\le r\exp\left(4\sum_{n=1}^{\infty}r^n\right) \\ &= r\exp\left(\frac{4r}{1-r}\right) \le d(0, \partial f(U)) \end{aligned}$$

if and only if

$$r\exp\left(\frac{4r}{1-r}\right) \le \frac{1}{e^2}.$$

Hence the Bohr radius, r_0 is the solution of

$$r\exp\left(\frac{4r}{1-r}\right) = \frac{1}{e^2}$$

which gives $r_0 \approx 0.09078$. Finally, it is evident that r_0 is attained by suitable rotations of the logharmonic Koebe function, f_0. □

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